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Exact scheme independence

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Abstract: Scheme independence of exact renormalization group equations, including independence of the choice of cutoff function, is shown to follow from general field redefinitions, which remains an inherent redundancy in quantum field theories. Renormalization group equations and their solutions are amenable to a simple formulation which is manifestly covariant under such a symmetry group. Notably, the kernel of the exact equations which controls the integration of modes acts as a field connection along the flow.

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Contents

1. Introduction

Quantum field theory provides a redundant tool to describe physics. A particular manifestation of this fact corresponds to the freedom to define a renormalization scheme in order for perturbative as well as non-perturbative computations to make explicit contact with observables. Independence of observables on different choices of scheme may be obscure if not impossible to check in practice, but it is a continuum property inherent to quantum field theory. In the framework of Wilson's Renormalization Group (RG) and in particular the exact RG [1], this property is especially deeply embedded and scheme independence is an issue that it is essential to address. Thus, a particular form of the exact RG receives its very definition in part by specifying a cutoff function. Beyond this, it is clear that one also has considerable further freedom in formulating flow equations [3]–[6]. This extra freedom in choosing a scheme can be put to great advantage [4]–[6]. We comment further below and in the conclusions.

The purpose of this note is to show that many changes of scheme in an exact RG equation can be implemented as field redefinitions. It is then clear that, within such a class of RGs, although the cutoff function and/or the form of the flow equation will vary, no universal properties will be modified; no observable prediction for the continuum quantum field theory will change. We shall proceed in three steps. First, we shall review what is already known and reanalyse Polchinski's exact RG equation in order to provide a simple understanding of its parts. Second, we shall perform a general field redefinition to show that its effect is to recast the original equation into one written in a different form. A change of cutoff function is seen to be just a particular case of the freedom to perform field redefinitions. Third, we take a broader view and show that the exact RG equation operates as a covariance statement. The final form and understanding of the exact RG equation will be very simple.

Note that there are very many ways to derive the exact RGs currently in use [1, 2, 4], [7]–[12]. Almost all of these derivations yield the same exact RG however, in some cases after some small transformations. (The underlying reason is that all such derivations start effectively by making the simplest choice of placing the effective cutoff Λ only in the bilinear kinetic terms.) Thus the Wegner-Houghton RG [2] is the sharp cutoff limit [12]–[14] of Polchinski's RG [9]. Polchinski's RG is transformed into Wilson's exact RG [1], by the momentum dependent change of variables $\varphi \mapsto \varphi\sqrt{K}$ that eliminates Polchinski's cutoff function K from the propagator [15].[1] Finally the exact RG for the Legendre effective action, or effective average action [8], [10]–[12], is the Legendre transform of Polchinski's equation [12, 16].

There have also been a number of studies of the dependence on the different forms of exact RG and on the form of the cutoff function within certain approximation schemes [17, 18].

The freedom in formulating exact RGs is more than this, as was originally pointed out by Wegner [3]. He noted that in principle a large class of exact RGs could be defined, based on three parts: the integrating out transformation, *viz.* "elimination of variables" (for some cases), the rescaling (or "dilatation", which may be incorporated by a change to scaling-dimensionless variables using Λ), and a possible infinitessimal field redefinition

$$\varphi_x \longmapsto \varphi_x + \delta\Lambda\, \Psi_x \tag{1.1}$$

under the small reduction $\Lambda \mapsto \Lambda - \delta\Lambda$. In fact, importantly, the first two parts may also be cast as field redefinitions. In this way, all the exact RGs correspond precisely to certain reparametrizations of the partition function [4].

We show this in section 2 by showing that rescaling is equivalent to a field redefinition. The fact that an integrating out transformation is also equivalent to a field redefinition then follows from differing derivations of the same exact RG [4, 12], and the equivalences between exact RGs recalled above [2]–[4].

In section 2 we concentrate on the Polchinski equation and establish that it implies precisely a reparametrization of the partition function under a scale dependent field redefinition. In section 3 we consider general field redefinitions and establish the general form and transformation properties of the resulting exact RGs. As an example we show that change of cutoff function in Polchinski's exact RG simply

[1] The physical meaning of the equivalence is not so clear to us however.

JHEP11(2000)004

corresponds to one of these field redefinitions. Appendix A contains some details for this computation and sketches a general method for finding such field redefinitions. In section 4 we explain how the transformations work when the t dependence is regarded as only inherited through the action. We recover the example of section 3 from this point of view. In section 5, we develop further the consequences of the resulting structures and show that they may all be simply understood in terms of a *connection* relating different scales. Finally in section 6 we draw our conclusions.

2. Revisiting Polchinski

To make the discussion concrete, consider Polchinski's form of the exact RG [9] written for the total effective action S, which we need in its renormalized scaled version [17]:[2]

$$\partial_t S + d\int_p \varphi_p \frac{\delta S}{\delta\varphi_p} + \int_p \varphi_p\, p^\mu \frac{\partial}{\partial p^\mu}\frac{\delta S}{\delta \varphi_p} = \int_p c'(p^2)\left(\frac{\delta S}{\delta\varphi_p}\frac{\delta S}{\delta\varphi_{-p}} - \frac{\delta^2 S}{\delta\varphi_p\delta\varphi_{-p}} - 2\frac{p^2}{c(p^2)}\varphi_p\frac{\delta S}{\delta\varphi_p}\right), \tag{2.1}$$

where $t \equiv \ln\frac{\mu}{\Lambda}$, μ being some fixed physical scale and Λ the Wilsonian cutoff which is taken towards 0, $S[\varphi, t]$ is a functional of φ and a function of t. On the l.h.s., the second term arises from rescaling the field, d being the full scaling dimension of the field, and the third term arises from rescaling the momenta (which is often rewritten in terms of a derivative that does not act on momentum-conserving δ functions [2] or as Δ_∂, the space-time derivative counting operator [15]). The scheme is defined through the choice of the cutoff function, which after scaling is just $c(p^2)$ (and thus $c'(p^2)$ means $\partial c(p^2)/\partial p^2$). It is less obvious at this level that the form of the r.h.s. is also a choice of scheme.

Actually, (2.1) does not arise from rescaling Polchinski's original formulation [9], unless we set d to the engineering dimension of the field (i.e. $d = D/2-1$ for a scalar field in spacetime dimension D) whereas we mean to incorporate also the anomalous dimension ($d(t) = D/2 - 1 + \gamma(t)/2$ for a scalar field) [14, 17]. Nevertheless, (2.1) is also a *perfectly valid* exact RG [4, 17] which can be justified on the grounds that γ can indeed be chosen as usual to constrain the normalisation of the kinetic term [17], that (2.1) still leaves the partition function invariant [14, 17] (as we will confirm below) and most importantly that (2.1) still corresponds to integrating out. (This latter follows on quite general grounds, because the partition function is left invariant as Λ decreases, but since all quantum corrections are UV regularised by Λ, i.e. limited above by Λ, the lost modes are being incorporated in S [5].) This is then one small but important example of the freedom allowed in the definition of the exact RGs.

[2] The unscaled version was discussed in this light in ref. [4].

JHEP11(2000)004

The exact RG equation (2.1) can also be written in a more enlightening form

$$\partial_t \, \mathrm{e}^{-S} = \partial_\alpha \left(\Psi^\alpha \mathrm{e}^{-S}\right), \tag{2.2}$$

where

$$\Psi^\alpha[\varphi, t] = (D - d(t))\,\varphi_p + p^\mu \frac{\partial \varphi_p}{\partial p^\mu} + c'(p^2)\left(-\frac{\delta}{\delta\varphi_{-p}} - 2\frac{p^2}{c(p^2)}\varphi_p\right), \tag{2.3}$$

discarding vacuum energy terms,[3] and we have introduced the compact notation

$$\partial_\alpha \equiv \frac{\delta}{\delta\varphi_p}, \tag{2.4}$$

(integration over p being implied by the summation convention. The sub- and super-index α can also stand for any further quantum number associated with the field φ.)

Note that (2.2) immediately implies that the partition function

$$\mathcal{Z} = \int \mathcal{D}\varphi \; \mathrm{e}^{-S} \tag{2.5}$$

is left invariant by (2.1), i.e. $\partial_t \mathcal{Z} = 0$, as claimed. Furthermore in this form, the r.h.s. of the exact RG equation corresponds to the integrand of a field redefinition [4], as we now explain in detail.

For an infinitessimal field redefinition $\tilde\varphi^\alpha = \varphi^\alpha - \theta^\alpha[\varphi, t]$ (optionally scale dependent) the jacobian is $|\delta\varphi/\delta\tilde\varphi| = 1 + \partial_\alpha\theta^\alpha$. Together with the term from the action itself, the changes in $\mathcal{Z}$ combine to produce the identity

$$\int \mathcal{D}\varphi \; \partial_\alpha \left(\theta^\alpha \mathrm{e}^{-S}\right) = 0\,. \tag{2.6}$$

Thus from (2.2), an infinitessimal step in t to $t + \delta t$, corresponds to the field redefinition

$$\tilde\varphi^\alpha = \varphi^\alpha - \delta t \; \Psi^\alpha[\varphi, t]\,. \tag{2.7}$$

This is in effect the way Polchinski's equation was originally deduced and brings separate meanings to both left and right hand sides of the exact RG equation. If integrated

$$0 = \partial_t \int \mathcal{D}\varphi \; \mathrm{e}^{-S} = \int \mathcal{D}\varphi \; \partial_\alpha \left(\Psi^\alpha \mathrm{e}^{-S}\right) = 0\,, \tag{2.8}$$

the l.h.s. vanishes because low-energy observables do not depend on the cut-off, whereas the r.h.s. is zero because it is the total derivative term emerging from a field redefinition as in (2.6).

Thus we may regard the underlying structure of an exact RG equation as a trade-off. A change of cut-off is compensated by a field redefinition.

[3] corresponding to the jacobian for scaling, and a term discarded in [9].

3. General field redefinitions

We are near to establishing the transformation properties of all the elements of (2.2) under field redefinitions. We shall now complete this analysis.

For the sake of precision let us start by recalling that the dependences of the kernel of the exact RG equation are

$$\Psi = \Psi^\alpha[\varphi, t]\,, \tag{3.1}$$

and that we want to analyse its behaviour under an infinitessimal field redefinition

$$\varphi^\alpha \longrightarrow \tilde{\varphi}^\alpha = \varphi^\alpha - \theta^\alpha[\varphi, t]\,. \tag{3.2}$$

On the one hand, we know that the integrand of the path integral $\exp(-S)$ transforms as a density on absorbing the jacobian of the field reparametrization:

$$\mathrm{e}^{-\tilde{S}} = \left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right| \mathrm{e}^{-S}\,. \tag{3.3}$$

Similarly we demand that the r.h.s. of (2.2) also transforms as a density i.e.

$$\tilde{\partial}_\alpha\left(\tilde{\Psi}^\alpha \mathrm{e}^{-\tilde{S}[\tilde{\varphi}]}\right) = \left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right| \partial_\alpha\left(\Psi^\alpha \mathrm{e}^{-S[\varphi]}\right). \tag{3.4}$$

This follows given (3.3), if we require that Ψ^α transforms as a vector, $\tilde{\Psi}^\alpha = \frac{\delta\tilde{\varphi}^\alpha}{\delta\varphi^\beta}\Psi^\beta$. On the other hand the l.h.s. transforms under a field redefinition as

$$\begin{aligned}
\partial_t|_{\tilde{\varphi}}\, \mathrm{e}^{-\tilde{S}[\tilde{\varphi}]} &= \left(\partial_t|_\varphi + \partial_t\varphi^\gamma|_{\tilde{\varphi}}\,\partial_\gamma\right)\left(\left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right| \mathrm{e}^{-S}\right) \\
&= \left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right| \partial_t|_\varphi\, \mathrm{e}^{-S} + \mathrm{e}^{-S}\,\partial_t|_\varphi \left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right| + \partial_t\varphi^\gamma|_{\tilde{\varphi}}\,\partial_\gamma\left(\left|\frac{\delta\varphi}{\delta\tilde{\varphi}}\right| \mathrm{e}^{-S}\right).
\end{aligned} \tag{3.5}$$

Substituting (2.2) and (3.4) for the first term, and expanding the other two to first order in θ^α,

$$\begin{aligned}
\partial_t|_{\tilde{\varphi}}\, \mathrm{e}^{-\tilde{S}} &= \tilde{\partial}_\alpha\left(\left(\tilde{\Psi}^\alpha + \partial_t\theta^\alpha\right)\mathrm{e}^{-\tilde{S}}\right) \\
&\equiv \tilde{\partial}_\alpha\left(\hat{\Psi}^\alpha \mathrm{e}^{-\tilde{S}}\right).
\end{aligned} \tag{3.6}$$

Thus, a field redefinition preserves the covariant form of the equation except for a shift in the functional form of the kernel. We may describe the change of Ψ just in terms of the original fields φ. Using the vector transformation law we get

$$\begin{aligned}
\delta\Psi^\alpha[\varphi, t] &\equiv \hat{\Psi}^\alpha[\varphi, t] - \Psi^\alpha[\varphi, t] \\
&= \partial_t\theta^\alpha + \theta^\beta\partial_\beta\Psi^\alpha - \Psi^\beta\partial_\beta\theta^\alpha\,.
\end{aligned} \tag{3.7}$$

JHEP11(2000)004

This can be further framed as a standard vector field transformation plus the explicit t derivative,

$$\delta\Psi^\alpha \overrightarrow{\partial_\alpha} = \partial_t\theta^\alpha \overrightarrow{\partial_\alpha} + \left[\theta^\beta \overrightarrow{\partial_\beta}, \Psi^\alpha \overrightarrow{\partial_\alpha}\right], \tag{3.8}$$

where we have introduced the notation $\overrightarrow{\partial_\alpha}$ to emphasise that here the ∂_α is understood to act on all possible terms appearing on its right.[4] A field redefinition modifies the kernel of the exact RG equation in this way. (Ref. [19] gives expressions for the transformation of the flow equation of the Legendre effective action, a.k.a. effective average action, under field redefinitions.)

Actually from (2.6) and (2.2), it is more appropriate for us to put the differential on the left. Let us define the following shorthand

$$\theta \equiv \overrightarrow{\partial_\alpha}\theta^\alpha\,, \qquad A_t \equiv \overrightarrow{\partial_\alpha}\Psi^\alpha\,, \qquad \delta A_t \equiv \overrightarrow{\partial_\alpha}\delta\Psi^\alpha \qquad \text{and} \qquad D_t \equiv \partial_t - A_t\,. \tag{3.9}$$

In this notation the exact RG (2.2) simply reads

$$D_t\, \mathrm{e}^{-S} = 0\,. \tag{3.10}$$

By (2.6), the change in the action under a field redefinition is just

$$\delta\, \mathrm{e}^{-S} = \theta\, \mathrm{e}^{-S} \tag{3.11}$$

and, either by integrating by parts the derivative in (3.8), or directly from (3.7),

$$\delta A_t = \partial_t\theta - [A_t, \theta] \;\equiv [D_t, \theta]\,. \tag{3.12}$$

Now it is easy to see that our transformation for Ψ^α does indeed ensure that the exact RG transforms covariantly:

$$\begin{aligned} \delta\left(D_t\, \mathrm{e}^{-S}\right) &= -\delta A_t\, \mathrm{e}^{-S} + D_t\, \delta\, \mathrm{e}^{-S} \\ &= -[D_t, \theta]\, \mathrm{e}^{-S} + D_t\, \theta\, \mathrm{e}^{-S} \\ &= \theta\left(D_t\, \mathrm{e}^{-S}\right). \end{aligned} \tag{3.13}$$

Let us illustrate the above discussion with the particular example of changing the cutoff in Polchinski's equation. For the moment we set $\gamma = 0$. The $\gamma \neq 0$ case is a little more involved and is solved in appendix A. First we note from (2.3), that acting on e^{-S} we may equivalently write

$$A_t \equiv \int_q \frac{\overrightarrow{\delta}}{\delta\varphi_q}\left((D-d)\,\varphi_q + q^\mu \frac{\partial}{\partial q^\mu}\varphi_q - c'(q^2)\frac{\overrightarrow{\delta}}{\delta\varphi_{-q}} - 2\frac{c'(q^2)}{c(q^2)}q^2\varphi_q\right). \tag{3.14}$$

We consider the infinitessimal field redefinition

$$\theta^\alpha = \frac{1}{2}\frac{\delta c(q^2)}{q^2}\frac{\delta S}{\delta\varphi_{-q}} - \frac{\delta c(q^2)}{c(q^2)}\varphi_q\,, \tag{3.15}$$

[4] up to eventually a functional on the far r.h.s. e.g. e^{-S}, which is here only implicit

which in (3.11), may equivalently be written

$$\theta \equiv -\frac{1}{2}\int_q \frac{\delta c(q^2)}{q^2}\frac{\overrightarrow{\delta}}{\delta\varphi_q}\left(\frac{\overrightarrow{\delta}}{\delta\varphi_{-q}}+2\frac{q^2}{c(q^2)}\varphi_q\right). \tag{3.16}$$

This can be derived by transforming the results of refs. [14]–[21] to the full action, however we construct it and for the first time the $\gamma \neq 0$ version, from first principles by a method described in appendix A.

Using (3.14) and (3.16), we have by the algebraic steps in (3.13), that (3.15) induces a change δA_t which acting on e^{-S} is equivalent to (3.12). Observing that $\partial_t\theta = 0$, and computing the commutator, one readily discovers that

$$\delta A_t \equiv -\int_q \frac{\overrightarrow{\delta}}{\delta\varphi_q}\left(\delta c'(q^2)\frac{\overrightarrow{\delta}}{\delta\varphi_{-q}}+2\left(\frac{\delta c'(q^2)}{c(q^2)}-c'(q^2)\frac{\delta c(q^2)}{c^2(q^2)}\right)q^2\varphi_q\right), \tag{3.17}$$

i.e. δA_t is just the change induced by $c \mapsto c+\delta c$ in (2.1). We have thus seen that any infinitessimal change of the function c in Polchinski's exact RG equation can be obtained through a field redefinition.

Let us mention that a change in $\gamma(t)$ (e.g. from zero to non zero) is also a field redefinition, whose form can be found similarly by the methods of appendix A.[5] Also note that a field redefinition induces a reparametrization in theory space, i.e. in the infinite dimensional coupling constant space that spans all effective actions. This latter viewpoint was pursued in ref. [14]. Here we see that such reparametrizations of theory space connect large classes of exact RGs, but we stress that fundamentally all this follows from field redefinitions.

4. Revealing the dependence in S

The above discussion on field redefinitions owns its simplicity to the convenient interpretation of the dependences of the kernel in the exact RG equation $\Psi^\alpha[\varphi,t]$. However, it is often the case that we want to think of it as a function of the action S. This is, for instance, the way we have first presented Polchinski's equation (2.1). There the field functional derivatives in $\Psi[\varphi,t]$ have acted on the $\exp(-S)$ and the simple structure advocated in (2.2) and (2.3) is hidden.

To avoid confusion we shall denote this new functionality with a bar notation,

$$\Psi^\alpha[\varphi,t]\mathrm{e}^{-S} = \bar{\Psi}[\varphi,S[\varphi,t]]\mathrm{e}^{-S}\,. \tag{4.1}$$

Because the φ functional derivatives have now acted explicitly on S, $\bar{\Psi}$ is just a function multiplying $\exp(-S)$. The kernel dependence on t is now coming only

[5] *despite* the fact that this maps between exact RGs with possibly strictly different fixed point structures

implicitly through the action. (Note that $\gamma(t)$ and S are not independent: one may be regarded as a function of the other through (2.1).) Of course, the functional forms Ψ and $\bar{\Psi}$ do transform differently under field redefinitions as we shall shortly show. It is arguable that, though natural, phrasing the dependences of the exact RG equation in terms of the action obscures its transformation properties.

We are now interested in equivalence under transformations whose t dependence is restricted in the following way:

$$\varphi \longrightarrow \tilde{\varphi}^\alpha = \varphi^\alpha - \bar{\theta}^\alpha[\varphi, S]\,. \tag{4.2}$$

Similarly, we have a new functionality for A_t which we denote by

$$\bar{A}_t \equiv \bar{A}_t[\varphi, S]\,, \qquad \bar{D}_t \equiv \partial_t - \bar{A}_t\,. \tag{4.3}$$

Upon field redefinitions, the transformation in the functional form of $\bar{A}_t$ carries a new piece

$$\delta \bar{A}_t[\varphi, S] \equiv \tilde{\bar{A}}_t[\varphi, S] - \bar{A}_t[\varphi, S] = [\bar{D}_t, \bar{\theta}] - \delta_S \bar{A}_t\,, \tag{4.4}$$

i.e. requires subtracting the induced change $\delta_S \bar{A}_t$. This new piece amounts to the simple chain rule

$$\delta_S \bar{A}_t = \int_\psi \left(\bar{\theta} S[\psi, t]\right) \frac{\delta \bar{A}_t}{\delta S[\psi, t]}\,. \tag{4.5}$$

It is straightforward to check that in this picture once again (3.10) is covariant: since the changes are described at constant S, (3.11) is missing in (3.13), but the new piece (4.5) in (4.4) supplies it instead — as can be seen by using the equation that follows from operating $\delta/\delta S[\psi, t]$ on (3.10).

Note that at a fixed point $S = S_*$, since we take $\bar{\theta}$ to depend on t only through S, $\bar{\theta}$ is also t independent. This means that fixed points $S_*^{(1)}$ are mapped to fixed points $S_*^{(2)}$. Furthermore, by the usual expansion in first order perturbations, the image fixed point $S_*^{(2)}$ has the same spectrum of eigenoperators as $S_*^{(1)}$. Therefore, at fixed points t-independent field redefinitions have no physical effects.

Two exact RGs are equivalent under field redefinitions if we can find a $\bar{\theta}$ such that we can map from one $\bar{A}_t[\varphi, S]$ to the other under (the exponential) of $\bar{\theta}$, and this in general will reduce to differential equations for the parts of $\bar{\theta}$. It is instructive to demonstrate (4.4) at a less formal level with the explicit transformation on $\bar{\Psi}^\alpha$ (again with $\gamma = 0$). Substituting (2.3) and (3.15) into (3.7), one finds a non-zero $\partial_t \bar{\theta}^\alpha = (\delta c(q^2)/2q^2)\, \delta \partial_t S/\delta\varphi_{-q}$, which is expanded via (2.1). Even after some manipulation, the result however is not equivalent to (3.17). This is because $\delta\bar{\Psi}^\alpha$ in (3.7) does not take into account the change induced via (3.3), of $\bar{\Psi}^\alpha$ as a functional of S. From (2.3), one needs to subtract from (3.7), the term $c'_p \frac{\delta}{\delta\varphi_{-p}} \delta S$, where from (3.11) and (3.15),

$$\delta S = \int_q \frac{\delta c(q^2)}{2q^2} \left(\frac{\delta S}{\delta\varphi_q} \frac{\delta S}{\delta\varphi_{-q}} - \frac{\delta^2 S}{\delta\varphi_q \delta\varphi_{-q}} - 2\frac{q^2}{c(q^2)} \varphi_q \frac{\delta S}{\delta\varphi_q} \right)\,. \tag{4.6}$$

(Incidentally the only restriction one has on such an infinitessimal δc is that the integral over q in this equation is bounded, in other words that δc decay fast enough for large momentum as required for a change in cutoff function.) Finally we find

$$
\begin{aligned}
\delta\bar{\Psi}^\alpha = \delta c'(p^2)\frac{\delta S}{\delta\varphi_{-p}} + 2\left(c'(p^2)\frac{\delta c(p^2)}{c^2(p^2)} - \frac{\delta c'(p^2)}{c(p^2)}\right) p^2\varphi_p + \\
+\frac{1}{2}\int_q \left(\frac{\delta c(p^2)}{p^2}c'(q^2) - \frac{\delta c(q^2)}{q^2}c'(p^2)\right)\left(\frac{\delta^2 S}{\delta\varphi_q\delta\varphi_{-p}}\frac{\delta S}{\delta\varphi_{-q}} - \frac{\delta^3 S}{\delta\varphi_{-q}\delta\varphi_q\delta\varphi_{-p}}\right),
\end{aligned}
\tag{4.7}
$$

which corresponds to the required change in (2.3), because the second line vanishes after substitution in (2.2), by symmetry.

5. The kernel Ψ^α as a connection

We have shown that invariance under field redefinitions allows mapping an exact RG equation into its version in a different scheme, as described by change of cutoff function. We now take a higher point of view and observe that field redefinitions further relate different functional forms of equally valid exact RG equations. Generically a field redefinition induced by a θ^α functional will alter the form of the equation, and implement highly non-trivial changes of scheme.

All the RG equations obtained in such a way belong to the same "universality" class, in the sense that no observable prediction for the continuum quantum field theory changes. It is apparent that exact RG equations can be transformed under field redefinitions, which correspond to a symmetry of the theory preserving low-energy observables. Working with a specific equation amounts to a choice for Ψ^α. The situation is reminiscent of the presence of local symmetries. Indeed this parallelism is already evident in our equations.

Thus, returning to a description in terms of a direct dependence on t, it is natural to interpret D_t in (3.10) as a covariant derivative, where from (3.9), the role of an antihermitian connection is played by A_t. Of course (3.12) is then nothing but a gauge transformation, carried by θ. Note that $\overrightarrow{\partial_\alpha}$ is the generator of reparametrizations (diffeomorphisms), i.e. the group we are dealing with. A_t is just the field Ψ^α contracted with the generators, i.e. valued in the Lie algebra. Similar remarks hold for θ and δA_t. We see from (3.11), that e^{-S} transforms homogeneously analogous to a matter field. And (3.13) merely confirms the now obvious gauge covariance of (3.10).

Concrete calculations at a given t need a choice of Ψ, that is A_t, within an orbit spanned by field redefinitions (carried out by θ). When moving to a lower cut-off, that is a different t, we are allowed to choose a different set of fields provided that a connection notifies the field redefinition which connect the two sets. The theory is field redefinition invariant, locally in t. Geometrically, the exact RG picks a covariant section through theory space (*a.k.a.* the space of actions) fibred over RG time.

However, the gauge theory interpretation is far from being just a formal viewpoint. Since this is one dimensional gauge theory, we know that any gauge field may at least locally be transformed into any other. In other words, locally in t, any exact RG can be transformed into any other (obviously however with the same fibres, i.e. field content). Furthermore, we can construct this transformation. We simply consider a Wilson line starting from say $t = t_0$:

$$\Phi(t) = P \exp\left(-\int_{t_0}^{t} ds\, A_s \right), \tag{5.1}$$

where P is path ordering, placing as usual the later As to the right. Thus $\partial_t \Phi(t) = -\Phi(t) A_t$ and $\partial_t \Phi^{-1}(t) = -A_t \Phi^{-1}(t)$. Therefore we have immediately,

$$A_t = -\Phi^{-1}(t)\, \partial_t\, \Phi(t)\,. \tag{5.2}$$

By (3.12), $\delta D_t = -[D_t, \theta]$, and thus (5.2) is a finite gauge transformation of $A_t = 0$. Since every gauge field is thus equivalent to $A_t = 0$, over any finite range in t, every gauge field is equivalent to each other over any finite range. Indeed by substituting (5.2) into (3.10), we obtain that any exact RG is equivalent to the trivial $\Psi^\alpha = 0$ equation,

$$\partial_t\, \mathrm{e}^{-S_0} = 0\,, \tag{5.3}$$

where

$$\mathrm{e}^{-S_0} = \Phi(t)\, \mathrm{e}^{-S}\,. \tag{5.4}$$

By (5.1), S_0 is nothing but S at $t = t_0$. Inverting this relation, we thus can write the solution of any exact RG in closed form as a series of integrals:

$$\mathrm{e}^{-S} = \Phi^{-1}(t)\, \mathrm{e}^{-S_0}\,. \tag{5.5}$$

It is easy to convince oneself that the momentum integrals obtained from expanding the notation in (5.5) converge, the result thus being a well-defined series for $\exp(-S)$.

Let us stress that the conclusion arrived at above, that every exact RG, i.e. A_t, is equivalent to every other over any finite range of t, holds when A_t is expressed as a function of φ and t only. Changing variables to φ and S in order to properly compare exact RGs, as in (4.1) and section 4, means that such an equivalence may then hold only over a limited domain or indeed break down entirely (for example at a fixed point), because of the difficulty of inverting this change of variables. There are also global obstructions to equivalence that can be understood from the gauge theory interpretation.

So far, we have been discussing non-observable quantities. A gauge field can be made to vanish at a particular point, but gauge-invariant observables retain its traces, and these are also obstructions to completely gauge away A_t. Generally, these obstructions can be either local or non local, but since the RG equation describes a one-dimensional flow in t, there are no local invariants which can be build from A_t. For instance, the equivalent of the field strength is identically zero in spite of the fact that the group of field redefinitions is non abelian. We are bound to use non-local invariants. Given that there are no flows that form closed loops in unitary Poincaré invariant theories[6] [22], the only remaining candidate is

$$\Phi = P \exp\left(-\int_{-\infty}^{\infty} ds\, A_s\right), \tag{5.6}$$

providing these limits exist. However, the existence of these limits implies that $A_t \to 0$ there, i.e. that we consider solutions of (2.2) that begin ($t = -\infty$) and end ($t = +\infty$) at fixed points. This quantity of course transforms homogeneously at both extremes of the flow with t-independent field redefinitions. To be more precise, the path-ordered line that connects the initial (i) fixed point with the final (f) one does transform under general field redefinitions as

$$\Phi_{fi} \longrightarrow \Omega_f \Phi_{fi} \Omega_i^{-1}, \tag{5.7}$$

where $\Omega_{i,f}$ are t-independent finite field redefinitions at the initial and final fixed points, respectively. Thus we have

$$\exp -S_*^f = \Phi_{fi} \exp -S_*^i\,. \tag{5.8}$$

Given that the two theories display different universality properties, it is then impossible to gauge away A_t since Φ^{fi} would then be unity modulo end-point field redefinitions. Equally since such a relation (5.8) transforms covariantly under field redefinitions, two exact RGs which are equivalent under field redefinitions must have the same network of flows between fixed points displaying the same spectrum of eigenoperators. And on the contrary, for two exact RGs that do not display equivalent fixed points with the same spectrum and network of flows, there can be no exponentiated solution θ mapping one to the other via (4.4).

We can rephrase the above argument in more physical terms. The inherent freedom to use field redefinitions allows for deforming at will the pace of integration of modes along the RG trajectory. It is always possible to completely stop the integration of modes along a finite part of the path. Nevertheless, the above obstruction to gauge away the connection means that modes will have to be integrated somewhere else. The extreme (unphysical) case would correspond to integrate all modes at a single scale. Looking back at Wilson's original deduction, it is clear that such freedom was present in the choice of scheme defined through his α function [1]. Our analysis is, though, more general than such a simple scheme change.

[6] corresponding to real euclidean invariant theories, in euclidean space.

Let us end up this section making clear the idea that field redefinitions play a double rôle in exact RG equations:

- The kernel Ψ^α of an exact RG equation is shaped as a field redefinition.
- A change of this kernel, i.e. of scheme, is implemented via a field redefinition.

The latter effect of field redefinitions is natural and shows intuitively that scheme changes are immaterial to physical observables. Instead, the first rôle may be confusing as it seems to go against the (irreversible) integration of modes picture underlying the RG. This paradox can be traced to the fact that A_t can be locally gauged away due to the 1-dimensional (that is in t) nature of the flow.

This element of confusion would be absent in other contexts, e.g. a standard 4-dimensional gauge theory where A_μ cannot be gauged away. There, A_μ carries physical particles as it is further reflected in the obstruction to get rid of the field strength $F_{\mu\nu}$. In the case of exact RG equations the very nature of the equation is attached to its kernel Ψ^α which cannot be gauged away globally as quantified by the t-ordered Wilson line Φ.

6. Conclusions

Exact renormalization group equations make quantitative the statement that low-energy physics is independent of the details of the cut-off. A specific change of that cut-off can be absorbed into a suitable field redefinition. This idea leads hierarchically to the standard Callan-Symanzik covariance statement in perturbation theory.

A particular exact RG equation is characterized by its kernel Ψ^α. The freedom to choose the form of this kernel is thus related to field redefinitions. Infinitessimally, two kernels, expressed in terms of the action S, are physically equivalent if they are related by (4.4). Globally, two kernels descibe the same RG flow between fixed points i and f if their corresponding connections A_1 and A_2 produce t-ordered lines which are related by

$$\Phi_{fi}[A_1] = \Omega_f \Phi_{fi}[A_2] \Omega_i^{-1} \,, \tag{6.1}$$

where $\Omega_{i,f}$ are t-independent field redefinitions of the fixed points actions at points i and f. In perturbation theory, an operator basis in chosen and all field redefinitions turn into finite coupling constant redefinitions. A change of perturbative scheme, that is of local counterterms, is tantamount to a finite redefinition of coupling constants.

Of course field redefinitions could be chosen which obscure the symmetries of the field theory. Equally, symmetries that are apparently broken by some exact RG, through its cutoff function or/and other parts of its structure, may only be deformed, rather than lost. This would be established if (the exponential of) a $\bar{\theta}$ satisfying (4.4) could be found that maps the exact RG to one preserving the symmetry.

JHEP11(2000)004

Freedom under field redefinition should be used for the practitioner's benefit. Though physics is independent of such redefinitions it is obvious that practical schemes do truncate in one way or another such freedom. Within a particular truncation, some *a priori* equivalent forms of exact RG equations will differ. Accurate results for observables may be approached faster in a particular renormalization scheme. Using the above results it is possible to construct a RG equation which carries higher functional derivatives than Polchinski's. Such an equation could be contrived so that its local potential approximation [23] is two-loop exact. Truncations, in particular this and higher orders in the derivative expansion, may well be more accurate than the corresponding results from the standard flow equations.

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JHEP11(2000)004

A. Finding the field redefinition corresponding to changing cutoff

If we restore a non-zero $\gamma(t)$, A_t may still be written equivalently as (3.14). We require to find $\theta[\varphi, t]$ such that formula (3.12) reproduces (3.17). It is helpful to introduce the operators:

$$
\begin{aligned}
\Delta^0[a] &= \int_p a(p^2) \frac{\overrightarrow{\delta}}{\delta\varphi_p} \varphi_p \,, \\
\Delta^1 &= \int_p \frac{\overrightarrow{\delta}}{\delta\varphi_p} p^\mu \frac{\partial}{p^\mu} \varphi_p \,, \\
\Delta^2[a] &= \int_p a(p^2) \frac{\overrightarrow{\delta}}{\delta\varphi_p} \frac{\overrightarrow{\delta}}{\delta\varphi_{-p}} \,.
\end{aligned} \tag{A.1}
$$

They form a closed algebra:

$$
\begin{aligned}
\left[\Delta^0[a], \Delta^0[b]\right] &= \left[\Delta^2[a], \Delta^2[b]\right] = 0 \,, \\
\left[\Delta^0[a], \Delta^2[b]\right] &= -2\Delta[ab] \,, \\
\left[\Delta^1, \Delta^0[a]\right] &= -2\Delta^0[p^2 a'] \,, \\
\left[\Delta^1, \Delta^2[a]\right] &= \Delta^2[Da - 2p^2 a'] \,.
\end{aligned} \tag{A.2}
$$

We may readily express (3.14) and (3.17) in terms of these. Taking as an ansatz

$$
\theta = \Delta^0 \left[H(p^2, t)\right] + \Delta^2 \left[K(p^2, t)\right] , \tag{A.3}
$$

we find from (3.12) and (A.2):

$$\frac{\partial}{\partial t} H + 2p^2 H' = -2p^2 \left(\frac{\delta c}{c}\right)'$$

$$\text{and} \qquad \frac{\partial}{\partial t} K + (2-\gamma) K + 2p^2 K' + 2Hc' - 4p^2 K c'/c = -\delta c' \,. \tag{A.4}$$

[Here $c \equiv c(p^2)$.] We take the simplest solution to the first equation: $H = -\delta c/c$, after which the second can be solved by the method of characteristics:

$$K(p^2, t) = -\frac{\delta c(p^2)}{2p^2} - \frac{c^2(p^2)}{2p^2}\, \mathrm{e}^{\Gamma(t)} \int^t \!ds\; \mathrm{e}^{-\Gamma(s)}\; \gamma(s)\, \frac{\delta c(p^2\, \mathrm{e}^{2s-2t})}{c^2(p^2\, \mathrm{e}^{2s-2t})}\,,$$

where $\Gamma(t) = \int^t ds\; \gamma(s)$. In the case that $\gamma = 0$, this solution collapses to (3.16).

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JHEP11(2000)004

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